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The remarkable algebra so*(2n), its representations, its Clifford algebra and potential applications

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Abstract. Properties of the real Lie algebra so*(2n) and its finite-dimensional representations are described. The structure of the Clifford algebras associated with the two fundamental spinor representations are determined, and it is found in particular that for so*(4n), there is a remarkable asymmetry: one representation is real, the other complex (though equivalent to its complex conjugate), and the associated Clifford algebras are inequivalent. Generalised Dirac matrices are defined, acting within the direct sum of these two spinor representations of so*(2n). These matrices generate a different Clifford algebra, not determined uniquely by those representations, and their commutators do not span a representation of so*(2n). The isomorphisms of so*(6) with su(3, 1), and of so*(8) with so(6, 2) are described, together with the corresponding mappings between finite-dimensional irreducible representations. The latter isomorphism is related to the triality property of so(8), and in particular maps the real fundamental spinor representation, the complex fundamental spinor representation, and the complex vector (defining) representation of so*(8), each of dimension eight, into the real vector representation and the two complex fundamental spinor representations of so(6, 2). More generally, some tensor representations of so*(8) are mapped into spinor, others into tensor representations of so(6, 2), and the same is true of spinor representations of so*(8). This raises the possibility of interesting applications of so*(2n), $n>4$, as a generalised spacetime symmetry or dynamical algebra, with tensor and spinor representations of so(3, 1) $<$ so*(2n) contained in an irreducible multiplet. Mention is made in particular of a recent proposal by Ward to base a unified field theory on so*(14) $>$ so(3, 1)$\oplus$su(3)$\oplus$su(2)$\oplus$u(1).

I. Introduction

The Lie groups SO(p, q) and their coverings have many important applications to physics, for various values of the non-negative integers p and q. We recall at once the spacetime groups [1] SO(3), SO(3, 1), etc, the use of SO(4, 2) as a dynamical group [2] and the use of the compact groups SO(p) to express symmetries in nuclear [3] and particle physics [4-6]. Accordingly, the representation theory of SO(p, q) has been explored fairly extensively [7, 8], especially for small values of p and q. In particular, the finite-dimensional irreducible representations (irreps) are well understood for all p and q, as is the structure of Clifford algebras [9-11] associated with the fundamental spinor irreps.

With $p+q=2n+1$, a fixed odd integer, the set of real Lie algebras so(p, q) exhausts the possible real forms of B_n, the complexification of so($2n+1$). However, if $p+q=2n$

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is even there is, for each n, one additional real form [8, 12–14] $so^*(2n)$ of D_n, the complexification of $so(2n)$. We know of no important applications of $so^*(2n)$ to physics until now, though the algebra has been mentioned in particle physics [15]; and although various aspects of the structure and representation theory have been developed—for example, ladder representations of $so^*(2n)$ have been constructed [15, 16], and $so^*(2n)$ has been used [17] in the construction of *models* [18] of the algebra $su(m)$—it is fair to say that $so^*(2n)$ remains unfamiliar to most mathematical and theoretical physicists.

Our present interest stems from a proposal by Ward [19] to use $so^*(14)$ as the basis for a unified gauge field theory of elementary particles. He has observed that since [8, 12–14]

$$\begin{aligned} so^*(8) &\simeq so(6,2) > so(3,2) \oplus su(2) \\ so^*(6) &\simeq su(3,1) > su(3) \oplus u(1) \end{aligned} \tag{1.1}$$

then

$$so^*(14) > so^*(8) \oplus so^*(6) > so(3,2) \oplus su(3) \oplus su(2) \oplus u(1). \tag{1.2}$$

The subalgebra $su(3) \oplus su(2) \oplus u(1)$ is of considerable interest in gauge theories, and we have here the possibility to combine it with the Lorentz algebra $so(3,1) < so(3,2)$.

Ward [19] has also noted the interesting feature that, because of the nature of the isomorphism between $so^*(8)$ and $so(6,2)$, an irrep of $so^*(2n)$, $n > 4$, will, in general, contain both tensor and spinor irreps of $so(3,1)$, so that there is a possibility of incorporating both bosons and fermions in an irreducible multiplet, with an associated $so^*(14)$-irreducible field.

In what follows we shall present some of the properties of $so^*(2n)$, with particular emphasis on the mapping between $so^*(8)$ and $so(6,2)$, and on the finite-dimensional irreps. The latter are determined by, and are in one-to-one correspondence with, those of $so(2n)$, which are well known. Nevertheless, there are several interesting special features, especially with regard to the reality of the fundamental spinor irreps, the structure of associated Clifford algebras, and the form of generalised Dirac matrices that link these two irreps.

It is not our purpose here to develop in detail any application of $so^*(2n)$ to particle physics, but in the final section we indicate briefly some interesting possibilities that provide the justification for our investigations.

II. Definitions and bases for so*(2n)

II.1. The group $SO^*(2n)$ can be identified [8] with that group of complex linear transformations g of C^{2n}, $z \to gz$, which satisfy

$$(gz)^{T}(gw) = z^{T}w \tag{2.1}$$

$$(gz)^{+}\eta(gw) = z^{+}\eta w \tag{2.2}$$

for all z, $w \in C^{2n}$, where

$$\eta = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \tag{2.3}$$

I_n being the $n\times n$ unit matrix (here and below T, $*$ and $\dagger$ denote transpose, complex conjugate and Hermitian conjugate, respectively). The matrices X in the corresponding Lie algebra are then defined by the conditions

$$X^{\mathrm{T}} = -X \tag{2.4a}$$

$$\eta X^{\dagger} = -X\eta. \tag{2.4b}$$

It follows from (2.1) and (2.2) that g has unit determinant, and hence from (2.1) that

$$\mathrm{SO}^*(2n) < \mathrm{SO}(2n, \mathrm{C}). \tag{2.5}$$

II.2. An alternative definition [14, 20] of $\mathrm{SO}^*(2n)$ is obtained by replacing (2.1) and (2.2) by

$$(gz)^{\mathrm{T}}\theta(gw) = z^{\mathrm{T}}\theta w \tag{2.6}$$

$$(gz)^{\dagger}\gamma(gw) = z^{\dagger}\gamma w \tag{2.7}$$

where

$$\theta = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \qquad \gamma = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}. \tag{2.8}$$

It can then be seen from (2.7) that

$$\mathrm{SO}^*(2n) < \mathrm{SU}(n, n). \tag{2.9}$$

The corresponding Lie algebra $\mathrm{so}^*(2n)$ now consists of matrices Y satisfying

$$Y^{\mathrm{T}} = -\theta Y\theta \tag{2.10a}$$

$$Y^{\dagger} = -\gamma Y\gamma. \tag{2.10b}$$

II.3. The isomorphism of the two groups or algebras defined in II.1 and II.2 is established with a unitary transformation which maps, in particular, a typical X into a typical Y:

$$Y = UXU^{\dagger} \qquad U = 2^{-1/2}\begin{bmatrix} I_n & -\mathrm{i}I_n \\ -\mathrm{i}I_n & I_n \end{bmatrix}. \tag{2.11}$$

II.4. We shall proceed from the second definition II.2, noting from (2.10) that in general Y has the form

$$Y = \begin{bmatrix} A+S & B+\mathrm{i}C \\ -B+\mathrm{i}C & A-S \end{bmatrix} \tag{2.12}$$

where A, B, C are $n\times n$ real antisymmetric matrices and S is an $n\times n$ pure imaginary symmetric matrix; all of A, B, C, S are therefore anti-Hermitian.

II.5. Since there are $\frac{1}{2}n(n-1)$ linearly independent matrices of type A, B or C and $\frac{1}{2}n(n+1)$ of type S, the dimension of the real Lie algebra $\mathrm{so}^*(2n)$ is $n(2n-1)$, as for $\mathrm{so}(2n)$.

II.6. We note from (2.12) the $\mathrm{u}(n)$ subalgebra of matrices with $B=C=0$, and the $\mathrm{so}(n)<\mathrm{u}(n)$ subalgebra with $B=C=S=0$.

II.7. It is easily seen from (2.6) and (2.7) with $n=1$, that

$$\mathrm{SO}^*(2) \simeq \mathrm{SO}(2) \tag{2.13}$$

so that $\mathrm{SO}^*(2)$ is compact. For $n>1$, $\mathrm{SO}^*(2n)$ and $\mathrm{so}^*(2n)$ are non-compact; for $n>2$ they are simple.

II.8. There exist the following isomorphisms [8, 12-14]:

$$\begin{aligned} &\mathrm{so}^*(2) \simeq \mathrm{so}(2) \\ &\mathrm{so}^*(4) \simeq \mathrm{su}(2) \oplus \mathrm{sl}(2, \mathrm{R}) \\ &\mathrm{so}^*(6) \simeq \mathrm{su}(3, 1) \\ &\mathrm{so}^*(8) \simeq \mathrm{so}(6, 2). \end{aligned} \tag{2.14}$$

II.9. For r, $s=1, 2, \ldots, n$, define

$$A_{rs} = E_{rs} - E_{sr} \qquad S_{rs} = \mathrm{i}(E_{rs} + E_{sr}) \tag{2.15}$$

where E_{rs} is the $n \times n$ matrix with 1 in the rth row, sth column, and zeros elsewhere, so that $E_{rs}E_{tu} = \delta_{st}E_{ru}$. Then setting

$$\begin{aligned} l_{rs} &= -l_{sr} = -\mathrm{i}\begin{bmatrix} A_{rs} & 0 \\ 0 & A_{rs} \end{bmatrix} \qquad & m_{rs} &= m_{sr} = -\mathrm{i}\begin{bmatrix} S_{rs} & 0 \\ 0 & -S_{rs} \end{bmatrix} \\ p_{rs} &= -p_{sr} = -\mathrm{i}\begin{bmatrix} 0 & A_{rs} \\ A_{sr} & 0 \end{bmatrix} \qquad & q_{rs} &= -q_{sr} = \begin{bmatrix} 0 & A_{rs} \\ A_{rs} & 0 \end{bmatrix} \end{aligned} \tag{2.16}$$

we see from (2.12) that $\mathrm{i}l_{rs}$, $\mathrm{i}p_{rs}$, $\mathrm{i}q_{rs}$ $(r>s)$ and $\mathrm{i}m_{rs}$ $(r \geqslant s)$ provide a basis for $\mathrm{so}^*(2n)$. From now on, we introduce the 'physicists' i' and refer to l_{rs}, p_{rs}, q_{rs} and m_{rs} as basis operators, allowing also r and s to range over $1, 2, \ldots, n$. The fundamental commutation relations of $\mathrm{so}^*(2n)$ are then

$$\begin{aligned} &[m_{rs}, m_{tu}] = \mathrm{i}(\delta_{st}l_{ru} - \delta_{rt}l_{us} - \delta_{ru}l_{ts} + \delta_{su}l_{rt}) \\ &[m_{rs}, p_{tu}] = \mathrm{i}(-\delta_{st}q_{ru} + \delta_{rt}q_{us} - \delta_{ru}q_{ts} + \delta_{su}q_{rt}) \\ &[m_{rs}, q_{tu}] = \mathrm{i}(\delta_{st}p_{ru} - \delta_{rt}p_{us} + \delta_{ru}p_{ts} - \delta_{su}p_{rt}) \\ &[p_{rs}, p_{tu}] = \mathrm{i}(\delta_{st}l_{ru} + \delta_{rt}l_{us} - \delta_{ru}l_{ts} - \delta_{su}l_{rt}) = [q_{rs}, q_{tu}] \\ &[p_{rs}, q_{tu}] = \mathrm{i}(-\delta_{st}m_{ru} + \delta_{rt}m_{us} - \delta_{ru}m_{ts} + \delta_{su}m_{rt}) \\ &[l_{rs}, x_{tu}] = \mathrm{i}(-\delta_{st}x_{ru} + \delta_{rt}x_{su} + \delta_{ru}x_{ts} - \delta_{su}x_{tr}) \end{aligned} \tag{2.17}$$

where x_{tu} denotes any of l_{tu}, m_{tu}, p_{tu}, q_{tu}. The l_{rs} span the $\mathrm{so}(n)$ subalgebra, and the l_{rs} and m_{rs} span the $\mathrm{u}(n)$ subalgebra, mentioned in II.6.

II.10. It is sometimes convenient instead to work with the complex combinations

$$\begin{aligned} &g^r_s = \tfrac{1}{2}(m_{rs} + \mathrm{i}l_{rs}) \\ &g^{rs} = -g^{sr} = \tfrac{1}{2}(p_{rs} - \mathrm{i}q_{rs}) \\ &g_{rs} = -g_{sr} = \tfrac{1}{2}(p_{rs} + \mathrm{i}q_{rs}) \end{aligned} \tag{2.18}$$

which satisfy the simpler relations

$$
\begin{aligned}
&[g^r_s, g^t_u] = \delta^t_s g^r_u - \delta^r_u g^t_s\\
&[g^r_s, g^{tu}] = \delta^t_s g^{ru} + \delta^u_s g^{tr}\\
&[g^r_s, g_{tu}] = -\delta^r_t g_{su} - \delta^r_u g_{ts}\\
&[g_{rs}, g^{tu}] = \delta^t_r g^u_s + \delta^u_s g^t_r - \delta^t_s g^u_r - \delta^u_r g^t_s\\
&[g_{rs}, g_{tu}] = 0 = [g^{rs}, g^{tu}]
\end{aligned}
\tag{2.19}
$$

and to bear in mind that the basis operators really consist of

$$
\begin{aligned}
&l_{rs} = -\mathrm{i}(g^r_s - g^s_r) \qquad && m_{rs} = g^r_s + g^s_r\\
&p_{rs} = g^{rs} + g_{rs} \qquad && q_{rs} = \mathrm{i}(g^{rs} - g_{rs}).
\end{aligned}
\tag{2.20}
$$

II.11. We find it useful at times also to bring out the relationship between so*($2n$) and so($2n$) by introducing the complex linear combinations j_{kl} $(=-j_{lk},\ k,\ l=1, 2, \dots, 2n)$ defined by

$$
\begin{aligned}
&j_{2r-1\ 2s-1} = \tfrac{1}{2}(-1)^{r+s}(l_{rs} - \mathrm{i}q_{rs})\\
&j_{2r-1\ 2s} = \tfrac{1}{2}(-1)^{r+s}(m_{rs} - \mathrm{i}p_{rs})\\
&j_{2r\ 2s} = \tfrac{1}{2}(-1)^{r+s}(l_{rs} + \mathrm{i}q_{rs})
\end{aligned}
\tag{2.21}
$$

for $r, s = 1, 2, \dots, n$. These satisfy the familiar so($2n$) relations

$$[j_{kl}, j_{mn}] = \mathrm{i}(\delta_{km}j_{ln} + \delta_{ln}j_{km} - \delta_{lm}j_{kn} - \delta_{kn}j_{lm}). \tag{2.22}$$

The relations inverse to (2.21) are

$$
\begin{aligned}
&l_{rs} = (-1)^{r+s}(j_{2r-1\ 2s-1} + j_{2r\ 2s})\\
&m_{rs} = (-1)^{r+s}(j_{2r-1\ 2s} + j_{2s-1\ 2r})\\
&p_{rs} = \mathrm{i}(-1)^{r+s}(j_{2r-1\ 2s} - j_{2s-1\ 2r})\\
&q_{rs} = \mathrm{i}(-1)^{r+s}(j_{2r-1\ 2s-1} - j_{2r\ 2s}).
\end{aligned}
\tag{2.23}
$$

II.12. In a representation of so*($2n$) we denote the representatives of l_{rs}, $m_{rs}, \dots, g^r_s$, $g^{rs}, \dots$, and j_{kl} by L_{rs}, M_{rs}, etc. If the representation is Hermitian (for example, one associated with a unitary representation of SO*($2n$)), then L_{rs}, M_{rs}, P_{rs} and Q_{rs} are Hermitian, while

$$
\begin{aligned}
&G^{r\dagger}_s = G^s_r \qquad && G^{rs\dagger} = G_{rs}\\
&J^\dagger_{2r-1\ 2s-1} = J_{2r\ 2s} \qquad && J^\dagger_{2r-1\ 2s} = J_{2s-1\ 2r}.
\end{aligned}
\tag{2.24}
$$

For $n > 1$, such a representation is either trivial or infinite-dimensional, since so*($2n$) is (semi-) simple and non-compact.

In a finite-dimensional representation, we can suppose without significant loss of generality that L_{rs} and M_{rs} are Hermitian, but that

$$P^\dagger_{rs} = -P_{rs} \qquad Q^\dagger_{rs} = -Q_{rs}. \tag{2.25}$$

Then

$$G^{r\dagger}_s = G^s_r \qquad G^{rs\dagger} = -G_{rs} \tag{2.26}$$

and all J_{kl} are Hermitian.

II.13. As II.11 shows, representations of so*(2n) and so(2n) are in one-to-one correspondence. It is convenient to display the basis operators J_{kl} of a representation of so(2n) in a triangular array,

$$\begin{array}{c|cc|cc|c}
J_{12} & J_{13} & J_{14} & J_{15} & J_{16} & \cdots \\
 & J_{23} & J_{24} & J_{25} & J_{26} & \\
\hline
 & & J_{34} & J_{35} & J_{36} & \cdots \\
 & & & J_{45} & J_{46} & \\
\hline
 & & & & J_{56} & \cdots \\
 & & & & & \vdots
\end{array} \tag{2.27}$$

In order to construct from this array, a set of basis operators for the corresponding representation of so*(2n), we proceed according to (2.23) and select firstly $2J_{12}$ $(=M_{11})$, $2J_{34}$ $(=M_{22})$, $2J_{56}$ $(=M_{33})$, ... etc. on the diagonal of the array. Then we choose from each 4×4 block $\boxed{\begin{matrix} J_{2r-1\,2s-1} & J_{2r-1\,2s} \\ J_{2r\,2s-1} & J_{2r\,2s}\end{matrix}}$ the combinations

$$(-1)^{r+s}(J_{2r-1\,2s-1}+J_{2r\,2s})\ (=L_{rs}) \qquad \mathrm{i}(-1)^{r+s}(J_{2r-1\,2s-1}-J_{2r\,2s})\ (=Q_{rs})$$
$$(-1)^{r+s}(J_{2r-1\,2s}-J_{2r\,2s-1})\ (=M_{rs}) \qquad \mathrm{i}(-1)^{r+s}(J_{2r-1\,2s}+J_{2r\,2s-1})\ (=P_{rs}).$$

This may be contrasted with the passage from basis operators of a representation of so(2n) to those of a corresponding representation of so($2n-m$, m), which can be achieved [8] by the Weyl trick, multiplying by i those elements contained in both the first $(2n-m)$ rows and last m columns of the array (2.27), and multiplying by (-1) the remaining elements of the last $(m-1)$ columns.

II.14. It is well known [8, 12] that the real forms of the complexification of a compact and simple real Lie algebra are defined by involutive automorphisms of the latter. In the case of so*(2n), the relevant involutive automorphism S of so(2n) is defined by

$$\begin{aligned} S: j_{2r\,2s} &\leftrightarrow j_{2r-1\,2s-1} \\ j_{2r\,2s-1} &\leftrightarrow j_{2s\,2r-1} \end{aligned} \tag{2.28}$$

for $r, s = 1, 2, \ldots, n$. According to a general prescription [8, 12], a basis for so*(2n) is then obtained from that for so(2n) by applying the transformation $P=\frac{1}{2}(1+\mathrm{i})I+\frac{1}{2}(1-\mathrm{i})S$, where I is the identity transformation. We get

$$\begin{aligned} P: j_{2r-1\,2s-1} &\rightarrow \tfrac{1}{2}(1+\mathrm{i})j_{2r-1\,2s-1}+\tfrac{1}{2}(1-\mathrm{i})j_{2r\,2s} && [=\tfrac{1}{2}(-1)^{r+s}(l_{rs}+q_{rs})] \\ j_{2r\,2s-1} &\rightarrow \tfrac{1}{2}(1+\mathrm{i})j_{2r\,2s-1}+\tfrac{1}{2}(1-\mathrm{i})j_{2s\,2r-1} && [=\tfrac{1}{2}(-1)^{r+s}(p_{rs}-m_{rs})] \\ j_{2r\,2s} &\rightarrow \tfrac{1}{2}(1+\mathrm{i})j_{2r\,2s}+\tfrac{1}{2}(1-\mathrm{i})j_{2r-1\,2s-1} && [=\tfrac{1}{2}(-1)^{r+s}(l_{rs}-q_{rs})]. \end{aligned} \tag{2.29}$$

It may be noted from (2.24) that S corresponds to the mapping of Hermitian conjugation in a Hermitian representation of so*(2n).

III. Finite-dimensional representations

III.1. An irrep of so(2n) is labelled completely [7] by a highest weight $(m_1, m_2, \ldots, m_n)$, corresponding to a choice of $(j_{12}, j_{34}, \ldots, j_{2n-1\,2n})$ as basis for a Cartan subalgebra. Here the m_r are either all integers or all half-odd integers, satisfying

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{n-1} \geqslant |m_n| \geqslant 0 \tag{3.1}$$

and each highest weight satisfying these conditions is allowed. We label the corresponding irrep of so*(2n) in the same way. Note that $(j_{12}, j_{34}, \ldots, j_{2n-1\,2n}) = (\frac{1}{2}m_{11}, \frac{1}{2}m_{22}, \ldots, \frac{1}{2}m_{nn})$.

III.2. The reality properties of the corresponding irreps of so*(2n) and so(2n) (or so(p, q), $p+q=2n$) differ. In particular, it is easily checked that the defining (2n-vector) representation (1, 0, 0, ..., 0) is pseudo-real (i.e. complex but equivalent to its complex conjugate) for so*(2n), though real for so(2n) (and for so(p, q)).

III.3. We turn now to the two fundamental spinor irreps $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2})$, each of dimension 2^{n-1}. In the case of so(2n), these two are contained in direct sum in the fundamental spinor irrep of so($2n+1$) > so(2n), of dimension 2^n. Labelling a basis J_{kl}, $J_{k\,2n+1}$ $(=-J_{2n+1\,k})$ for an irrep of so($2n+1$) in the usual way, with $k, l=1, 2, \ldots, 2n$, we exhibit this basis as a triangular array generalising that for so(2n) in (2.27); it has one extra column on the right with elements $J_{k\,2n+1}$. A basis for the fundamental spinor irrep of so($2n+1$) is given in terms of n commuting sets of Pauli matrices [22] $\sigma_\alpha, \rho_\alpha, \tau_\alpha, \ldots$ ($\alpha = 1, 2, 3$). Thus for so(3) and so(5) the triangular arrays are (apart from an overall factor of $\frac{1}{2}$)

$$\begin{array}{cc} \sigma_3 & -\sigma_2 \\ & \sigma_1 \end{array} \quad \text{and} \quad \begin{array}{cccc} \sigma_3 & -\sigma_2 & \sigma_1\rho_3 & -\sigma_1\rho_2 \\ & \sigma_1 & \sigma_2\rho_3 & -\sigma_2\rho_2 \\ & & \sigma_3\rho_3 & -\sigma_3\rho_2 \\ & & & \rho_1 \,. \end{array} \tag{3.2}$$

The pattern of successive arrays generalising these two can be developed as follows. In going from so($2n-1$) to so($2n+1$) we introduce a new set of Pauli matrices ω_α, say. Suppose that the corresponding set introduced at the previous stage (going from so($2n-3$) to so($2n-1$)) was ν_α. We add two columns on the right of the array for so($2n-1$). The first is obtained by multiplying the last column of the so($2n-1$) array by $-\mathrm{i}\nu_3\omega_3$ on the left, and adding an element $\nu_3\omega_3$ at the bottom. The second is obtained by multiplying the first additional column by $-\mathrm{i}\omega_1$ on the left, and adding an element ω_1 at the bottom. For example, the third and fourth columns of the so(5) array in (3.2) are obtained in this way from the so(3) array, with multiplications by $-\mathrm{i}\sigma_3\rho_3$ and then $-\mathrm{i}\rho_1$.

More generally, we build up in this way the array

$$\begin{array}{ccccccccc} \sigma_3 & -\sigma_2 & \sigma_1\rho_3 & -\sigma_1\rho_2 & \sigma_1\rho_1\tau_3 & -\sigma_1\rho_1\tau_2 & \sigma_1\rho_1\tau_1\lambda_3 & -\sigma_1\rho_1\tau_1\lambda_2 & \ldots \\ & \sigma_1 & \sigma_2\rho_3 & -\sigma_2\rho_2 & \sigma_2\rho_1\tau_3 & -\sigma_2\rho_1\tau_2 & \sigma_2\rho_1\tau_1\lambda_3 & -\sigma_2\rho_1\tau_1\lambda_2 & \ldots \\ & & \sigma_3\rho_3 & -\sigma_3\rho_2 & \sigma_3\rho_1\tau_3 & -\sigma_3\rho_1\tau_2 & \sigma_3\rho_1\tau_1\lambda_3 & -\sigma_3\rho_1\tau_1\lambda_2 & \ldots \\ & & & \rho_1 & \rho_2\tau_3 & -\rho_2\tau_2 & \rho_2\tau_1\lambda_3 & -\rho_2\tau_1\lambda_2 & \ldots \\ & & & & \rho_3\tau_3 & -\rho_3\tau_2 & \rho_3\tau_1\lambda_3 & -\rho_3\tau_1\lambda_2 & \ldots \\ & & & & & \tau_1 & \tau_2\lambda_3 & -\tau_2\lambda_2 & \ldots \\ & & & & & & \tau_3\lambda_3 & -\tau_3\lambda_2 & \ldots \\ & & & & & & & \lambda_1 & \ldots \end{array} \tag{3.3}$$

of which the array for so($2n+1$) consists of the first $2n$ columns.

The triangular arrays for the fundamental spinor irreps of $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2})$ of so(2n) are obtained from the array for the so($2n+1$) spinor by deleting the last column, and replacing ω_3 by its eigenvalues ± 1 in turn. For example, for so(4) we get from the so(5) array (3.2),

$$\begin{array}{ccc} \sigma_3 & -\sigma_2 & \sigma_1\rho_3 \\ & \sigma_1 & \sigma_2\rho_3 \\ & & \sigma_3\rho_3 \end{array} \qquad \text{and then} \qquad \begin{array}{ccc} \sigma_3 & -\sigma_2 & \pm\sigma_1 \\ & \sigma_1 & \pm\sigma_2 \\ & & \pm\sigma_3\,. \end{array} \tag{3.4}$$

The basis operators of the fundamental spinor irreps of so*$(2n)$ can then be constructed as in II.13. For the case of so*(4) we get from (3.4)

$$\begin{aligned} &M_{11}=\sigma_3 \qquad M_{22}=\pm\sigma_3 \qquad L_{12}=\tfrac{1}{2}(\sigma_2\mp\sigma_2) \\ &Q_{12}=\tfrac{1}{2}\mathrm{i}(\sigma_2\pm\sigma_2) \qquad M_{12}=\tfrac{1}{2}(\sigma_1\mp\sigma_1) \qquad P_{12}=-\tfrac{1}{2}\mathrm{i}(\sigma_1\pm\sigma_1). \end{aligned} \tag{3.5}$$

The only non-zero basis operators for the irrep $(\frac{1}{2},\frac{1}{2})$ are therefore σ_3, $\mathrm{i}\sigma_2$ and $\mathrm{i}\sigma_1$, while for the irrep $(\frac{1}{2},-\frac{1}{2})$ they are σ_3, σ_2 and σ_1. We see that $(\frac{1}{2},\frac{1}{2})$ can be regarded essentially as a two-dimensional irrep of so(2, 1) $\simeq$ sl(2, R), and is therefore (equivalent to) a real irrep, while $(\frac{1}{2},-\frac{1}{2})$ is essentially a two-dimensional irrep of su(2) and is therefore pseudo-real. The association of the two spinors with different subalgebras is not typical of higher values of n, and reflects the fact that so*(4) $\simeq$ sl(2, R)$\oplus$su(2) is not simple. However, the fact that one spinor irrep is real and the other is pseudo-real is typical of so*$(4n)$, $n=1, 2, \ldots$, as we shall see.

III.4. The fundamental spinor irreps of so*$(2n)$, so$(2n)$ and so$(2n+1)$ can also be defined [23] concisely in terms of n Fermi creation operators a^r and conjugate annihilation operators a_r, $r=1, 2, \ldots, n$ satisfying

$$\{a_r, a^s\}=\delta_r^s I \qquad \{a_r, a_s\}=\{a^r, a^s\}=0. \tag{3.6}$$

Such an algebra is well known to have an irrep of dimension 2^n; I is the corresponding unit operator. For so$(2n)$ and so*$(2n)$ we set

$$\begin{aligned} &G^r_s=\tfrac{1}{2}[a^r, a_s]=a^ra_s-\tfrac{1}{2}\delta^r_sI \qquad G^{rs}=\tfrac{1}{2}[a^r, a^s]=a^ra^s \\ &G_{rs}=\tfrac{1}{2}[a_r, a_s]=a_ra_s. \end{aligned} \tag{3.7}$$

Corresponding L_{rs}, $M_{rs}, \ldots$ and J_{kl} can then be defined as in (2.20) and (2.21). For so$(2n+1)$ we add also $J_{k\;2n+1}$ $(=-J_{2n+1\;k})$, $k=1, 2, \ldots, 2n$, defined by

$$J_{2r-1\;2n+1}=-\tfrac{1}{2}(a_r+a^r) \qquad J_{2r\;2n+1}=-\tfrac{1}{2}\mathrm{i}(a_r-a^r) \tag{3.8}$$

with $r=1, 2, \ldots, n$.

We introduce a *vacuum vector* $|0\rangle$ with $a_r|0\rangle=0$, $r=1, 2, \ldots, n$. The carrier space for one spinor irrep of so*$(2n)$ and so$(2n)$ is spanned by the vectors

$$|0\rangle,\ a^ra^s|0\rangle,\ a^ra^sa^ta^u|0\rangle, \ldots \tag{3.9}$$

corresponding to even eigenvalues $0, 2, 4, \ldots, 2m$ of the *number operator*

$$N=a^ra_r=G^r_r+\tfrac{1}{2}nI=\tfrac{1}{2}(M_{11}+M_{22}+\ldots+M_{nn})+\tfrac{1}{2}nI \tag{3.10}$$

where $2m=n$ or $n-1$, according as n is even or odd. (Recall that a product of more than n creation operators vanishes.) The number of linearly independent vectors in (3.9) is $\binom{n}{0}+\binom{n}{2}+\ldots+\binom{n}{2m}=2^{n-1}$ as required.

Similarly, the carrier space for the other spinor irrep is spanned by

$$a^r|0\rangle,\ a^ra^sa^t|0\rangle, \ldots \tag{3.11}$$

corresponding to odd eigenvalues $1, 3, \ldots, 2m+1$ of N, where $2m+1=n-1$ or n, according as n is even or odd; again there are $\binom{n}{1}+\binom{n}{3}+\ldots+\binom{n}{2m+1}=2^{n-1}$ linearly independent vectors here. Since (3.10) implies that N has the eigenvalue n on a vector with weight $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})$, it follows that the irrep $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})$ corresponds to (3.9) if n is even and to (3.11) if n is odd. The vectors (3.9) and (3.11) together span the whole 2^n-dimensional space, carrying the fundamental spinor irrep of so$(2n+1)$.

III.5. The equivalence of the realisations described in III.3 and III.4 is established by setting (for $n>1$)

$$\begin{aligned}
a^1 &= \tfrac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2)\rho_1\tau_1 \ldots \mu_1\nu_1\omega_2 & a_1 &= \tfrac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2)\rho_1\tau_1 \ldots \mu_1\nu_1\omega_2\\
a^2 &= \tfrac{1}{2}(\sigma_3\rho_1 + \mathrm{i}\rho_2)\tau_1 \ldots \mu_1\nu_1\omega_2 & a_2 &= \tfrac{1}{2}(\sigma_3\rho_1 - \mathrm{i}\rho_2)\tau_1 \ldots \mu_1\nu_1\omega_2\\
a^3 &= \tfrac{1}{2}(\rho_3\tau_1 + \mathrm{i}\tau_2) \ldots \mu_1\nu_1\omega_2 & a_3 &= \tfrac{1}{2}(\rho_3\tau_1 - \mathrm{i}\tau_2) \ldots \mu_1\nu_1\omega_2\\
&\vdots & &\vdots\\
a^{n-1} &= \tfrac{1}{2}(\mu_3\nu_1 + \mathrm{i}\nu_2)\omega_2 & a_{n-1} &= \tfrac{1}{2}(\mu_3\nu_1 - \mathrm{i}\nu_2)\omega_2
\end{aligned}$$

and

$$a^n = \tfrac{1}{2}(\nu_3\omega_2 - \mathrm{i}\omega_1) \qquad a_n = \tfrac{1}{2}(\nu_3\omega_2 + \mathrm{i}\omega_1). \tag{3.12}$$

(For $n=1$, we simply set $a^1 = \frac{1}{2}(\sigma_2 - \mathrm{i}\sigma_1)$, $a_1 = \frac{1}{2}(\sigma_2 + \mathrm{i}\sigma_1)$.)

Note that

$$\begin{aligned}
&a^1a_1 = \tfrac{1}{2}(I+\sigma_3),\ a^2a_2 = \tfrac{1}{2}(I+\sigma_3\rho_3), \ldots, a^na_n = \tfrac{1}{2}(I+\nu_3\omega_3)\\
&N = \tfrac{1}{2}(\sigma_3 + \sigma_3\rho_3 + \rho_3\tau_3 + \ldots + \nu_3\omega_3) + \tfrac{1}{2}nI.
\end{aligned} \tag{3.13}$$

III.6. In these spinor representations as constructed, the Hermiticity relations (2.25) and (2.26) hold. Since L_{rs} and M_{rs} commute with N, while P_{rs} and Q_{rs} consist of linear combinations of operators which shift the value of N up or down by two units, as (3.7) and (2.20) show, we have

$$[\zeta, L_{rs}] = [\zeta, M_{rs}] = \{\zeta, P_{rs}\} = \{\zeta, Q_{rs}\} = 0 \tag{3.14}$$

where

$$\zeta = \exp(\mathrm{i}\pi N/2). \tag{3.15}$$

Equivalently

$$\zeta Z^{\dagger} = Z\zeta \tag{3.16}$$

where Z is any of the so*$(2n)$ basis operators L_{rs}, M_{rs}, P_{rs}, Q_{rs}. On the vectors (3.9), ζ has eigenvalues ± 1, so $\zeta^2 = I$ in the corresponding spinor irrep $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, (-1)^n\frac{1}{2})$. On the vectors (3.11), ζ has eigenvalues $\pm\mathrm{i}$, so that $\zeta^2 = -I$ in the spinor irrep $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, (-1)^{n+1}\frac{1}{2})$. In any case, (3.16) shows that ζ is essentially a *Hermitising operator* for these spinor irreps. Note from (3.13) that

$$\begin{aligned}
\zeta &= \mathrm{e}^{\mathrm{i}\pi n/4} \exp(\mathrm{i}\pi\sigma_3/4)\exp(\mathrm{i}\pi\sigma_3\rho_3/4) \ldots \exp(\mathrm{i}\pi\nu_3\omega_3/4)\\
&= [\tfrac{1}{2}(1+\mathrm{i})]^n(1+\mathrm{i}\sigma_3)(1+\mathrm{i}\sigma_3\rho_3)\ldots(1+\mathrm{i}\nu_3\omega_3).
\end{aligned} \tag{3.17}$$

IV. Clifford algebras and generalised Dirac matrices

IV.1. We begin by recalling some results for Dirac matrices and Clifford algebras [9-11] associated with the fundamental spinor representations of so(p, q), $p>0$, $q \geqslant 0$, $p+q=2n$. We start with the triangular array for the fundamental spinor irrep of so$(2n+1)$, and (for $q>0$) multiply by i all elements J_{kl}, $J_{k\,2n+1}$ with k or l (but not both) $\in \{2n, 2n-1, \ldots, 2n+1-q\}$ and multiply by (-1) all elements J_{kl} with both k and l in that set. The operators in the array so defined provide a set of basis operators J'_{kl}, $J'_{k\,2n+1}$ of the fundamental spinor irrep of so$(p+1, q)$. The elements of the last column are generalised Dirac matrices γ'_k $(=2J'_{k\,2n+1})$ for so(p, q), satisfying

$$\begin{aligned} &\{\gamma'_k, \gamma'_l\} = 2g_{kl}I \qquad \tfrac{1}{4}[\gamma'_k, \gamma'_l] = \mathrm{i}J'_{kl} \\ &\mathrm{i}[\gamma'_k, J'_{lm}] = g_{kl}\gamma'_m - g_{km}\gamma'_l \end{aligned} \tag{4.1}$$

where $(g_{kl}) = \mathrm{diag}(1, 1, \ldots, 1, -1, \ldots, -1)$ [$p(+1)$s and $q(-1)$s].

The last of the relations (4.1) shows that the γ'_k form a $2n$-vector operator with respect to so(p, q). We also introduce

$$\varepsilon' = \gamma'_1\gamma'_2 \ldots \gamma'_{2n} = (\mathrm{i})^{n+q}\omega_3 \qquad (\varepsilon')^2 = (-1)^{n+q}I = (-1)^{(p-q)/2}I. \tag{4.2}$$

The operators J'_{kl} are basis operators for (the direct sum of) the two fundamental spinor irreps of so(p, q), corresponding to the eigenvalues ± 1 of ω_3, and hence $\pm(\mathrm{i})^{n+q}$ of ε'. The γ'_k are intertwining operators for these two irreps:

$$\{\gamma'_k, \varepsilon'\} = 0. \tag{4.3}$$

For example, to go from so(5) to so(3, 2) and then to so(2, 2), we replace the array (3.2) by

$$\begin{matrix} \sigma_3 & -\mathrm{i}\sigma_2 & \mathrm{i}\sigma_1\rho_3 & -\sigma_1\rho_2 \\ & \mathrm{i}\sigma_1 & \mathrm{i}\sigma_2\rho_3 & -\sigma_2\rho_2 \\ & & -\sigma_3\rho_3 & -\mathrm{i}\sigma_3\rho_2 \\ & & & \mathrm{i}\rho_1 \end{matrix} \quad = \quad \begin{matrix} 2J'_{12} & 2J'_{13} & 2J'_{14} & \gamma'_1 \\ & 2J'_{23} & 2J'_{24} & \gamma'_2 \\ & & 2J'_{34} & \gamma'_3 \\ & & & \gamma'_4 . \end{matrix} \tag{4.4}$$

As a consequence of the first of (4.1), the γ'_l generate the real Clifford algebra $\mathscr{C}(p, q)$ with even part $\mathscr{C}^+(p, q)$ generated by the operators $\mathrm{i}J'_{kl}$, and, more specifically, by the operators $\xi'_k = \mathrm{i}J'_{1k}$, $k = 2, 3, \ldots, 2n$ which satisfy

$$\{\xi'_k, \xi'_l\} = -2g_{kl}I \qquad k, l = 2, 3, \ldots, 2n. \tag{4.5}$$

Thus $\mathscr{C}^+(p, q) \simeq \mathscr{C}(q, p-1)$.

IV.2. The algebras $\mathscr{C}(p, q)$ have been classified and their structure analysed [9-11]. As we have specified here that $p+q=2n$ is even, we know that $\mathscr{C}(p, q)$ is simple. Its centre, generated by I, is isomorphic to R, the algebra of real numbers. Furthermore, we have

$$\mathscr{C}(p, q) \simeq \begin{cases} \mathscr{M}(2^n, \mathrm{R}) & p-q = 0 \text{ or } 2 \ (\mathrm{mod.}\ 8) \\ \mathscr{M}(2^{n-1}, \mathrm{H}) & p-q = 4 \text{ or } 6 \ (\mathrm{mod.}\ 8) \end{cases} \tag{4.6}$$

where $\mathscr{M}(2^n, \mathrm{R})$ and $\mathscr{M}(2^{n-1}, \mathrm{H})$ are the algebras of $2^n \times 2^n$ real, and $2^{n-1} \times 2^{n-1}$ quaternionic matrices, respectively.

The subalgebra $\mathscr{C}^+(p,q)$ has a non-trivial centre, generated by I and ε'. (Since ε' anticommutes with each γ'_k, it commutes with any product of an even number of the γ'_k, and so with all *even* elements of $\mathscr{C}(p,q)$.) If $p-q=2$ (mod. 4), then $(p-q)/2$ is odd and (4.2) shows that $(\varepsilon')^2=-I$. Then $\mathscr{C}^+(p,q)$ is simple, with centre isomorphic to C, the algebra of complex numbers; we have in fact

$$\mathscr{C}^+(p,q)\simeq\mathscr{C}(q,p-1)\simeq\mathscr{M}(2^{n-1},\mathrm{C})\qquad p-q=2\,(\mathrm{mod.}\,4). \tag{4.7}$$

If $p-q=0$ (mod. 4), then $(p-q)/2$ is even, $(\varepsilon')^2=+I$, and the centre of $\mathscr{C}^+(p,q)$ is isomorphic to R$\oplus$R. Then $\mathscr{C}^+(p,q)$ is not simple, but splits into a (direct) sum of two ideals:

$$\mathscr{C}^+(p,q)=\mathscr{C}^+(p,q)\tfrac{1}{2}(I-\varepsilon')\oplus\mathscr{C}^+(p,q)\tfrac{1}{2}(I+\varepsilon'). \tag{4.8}$$

Each ideal is isomorphic to $\mathscr{M}(2^{n-1},\mathrm{R})$ if $p-q=0$ (mod. 8), and to $\mathscr{M}(2^{n-2},\mathrm{H})$ if $p-q=4$ (mod. 8), i.e.

$$\mathscr{C}^+(p,q)\simeq\mathscr{C}(q,p-1)\simeq\begin{cases}\mathscr{M}(2^{n-1},\mathrm{R})\oplus\mathscr{M}(2^{n-1},\mathrm{R}) & p-q=0\,(\mathrm{mod.}\,8)\\ \mathscr{M}(2^{n-2},\mathrm{H})\oplus\mathscr{M}(2^{n-2},\mathrm{H}) & p-q=4\,(\mathrm{mod.}\,8).\end{cases} \tag{4.9}$$

The two ideals in each case are associated with the eigenvalues ±1 of ε' and hence with the two fundamental spinor irreps of so(p,q).

IV.3. Since these irreps are of dimension 2^{n-1}, it follows from (4.7) and (4.9) that they are real representations if and only if $p=q$ (mod. 8). On the other hand, it is easily seen from a consideration of weights that each of these irreps is (at least) pseudo-real if $(p+q)/2$ $(=n)$ and q are both even or both odd. If one is even and the other odd, then the two spinor irreps are complex conjugate to each other.

IV.4. Contragradience relations [21] amongst irreps are the same for all real forms of D_n. In particular, the two fundamental spinor irreps are contragradient to each other if n is odd, and self-contragradient if $n=2m$ is even. (In this latter case they are orthogonal or symplectic according as m is even or odd.)

IV.5. We now turn to the case of so*$(2n)$ beginning again with the 2^n-dimensional space carrying the direct sum of the two fundamental spinor irreps, and with the associated array of basis operators J_{kl} of so$(2n)$, expressed as before in terms of either n sets of Pauli matrices or n pairs of Fermi operators. Dirac matrices γ_a, $a=1,2,\ldots,2n$ must now be required to transform as a $2n$-vector operator with respect to so*$(2n)$. Thus if L_{rs}, $M_{rs}\ldots$ are the so*$(2n)$ basis operators in the 2^n-dimensional space, we must have

$$[\gamma_a,L_{rs}]=(l_{rs})_{ab}\gamma_b\qquad[\gamma_a,M_{rs}]=(m_{rs})_{ab}\gamma_b\qquad\text{etc} \tag{4.10}$$

where l_{rs} etc. are basis operators in (a representation equivalent to) the defining ($2n$-vector) representation, with matrix elements $(l_{rs})_{ab}$ etc. Here $r,s=1,2,\ldots,n$ and a, $b=1,2,\ldots,2n$, and the repeated indices in (4.10) are summed over. (We use a, $b,\ldots$ rather than k, $l,\ldots$ here, as the former are not necessarily tensor indices with respect to so$(2n)$ operators J_{kl}, as we shall see.)

It follows from (4.10) that

$$[\{\gamma_a,\gamma_b\},L_{rs}]=(l_{rs})_{ac}\{\gamma_c,\gamma_b\}+(l_{rs})_{bc}\{\gamma_a,\gamma_c\}\qquad\text{etc.} \tag{4.11}$$

If we are to have

$$\{\gamma_a,\gamma_b\}=2\theta_{ab}I \tag{4.12}$$

for some real, non-singular, symmetric $2n \times 2n$ matrix θ with elements θ_{ab}, then (4.11) implies

$$l_{rs}\theta + \theta l_{rs}^{\mathrm{T}} = 0 \qquad \text{etc.} \tag{4.13}$$

With l_{rs} etc as in (2.16), this is satisfied only with θ as in (2.8), (2.10a), up to an arbitrary multiplicative constant that can be set to unity by scaling the γ_a's.

Can we find γ_a satisfying (4.10) and (4.12)? The answer is yes: it is not hard to check that, in terms of the Fermi operators (3.6), we can set

$$\gamma_r = (1-\mathrm{i})a_r \qquad \gamma_{n+r} = (1+\mathrm{i})a^r \qquad r = 1, 2, \ldots, n. \tag{4.14}$$

Their expression in terms of Pauli matrices is then determined by (3.12). It is easy to see that they are intertwining operators for the two fundamental spinor irreps of so*$(2n)$; they anticommute with the last of the Pauli matrices ω_3, which labels these irreps (see also (4.16) and (4.17) below).

Because θ can be brought to the diagonal form γ as in (2.8), by a real orthogonal transformation, it follows that the real Clifford algebra generated by the γ_a is isomorphic to $\mathscr{C}(n, n) \simeq \mathscr{M}(2^n, \mathrm{R})$. However, the Lie algebra spanned by the commutators $[\gamma_a, \gamma_b]$ is evidently isomorphic to so(n, n), not so*$(2n)$; the latter is not included in $\mathscr{C}(n, n)$ in the way that so(p, q) is included in $\mathscr{C}(p, q)$, as a subset of (even) elements that are quadratic in the Dirac matrices.

Since the l_{rs}, m_{rs}, etc. in (4.10) need only be *equivalent* to our operators in (2.16), we can more generally choose them to have the form $l'_{rs} = S l_{rs} S^{-1}$ etc., with l_{rs} etc as in (2.16) and S any complex non-singular $2n \times 2n$ constant matrix. Then θ in (4.12) is replaced by $\theta' = S\theta S^{\mathrm{T}}$. Since S can be complex, it is possible to make $\theta' = I_{2n}$, so that $\{\gamma_a, \gamma_b\} = 2\delta_{ab} I$; indeed, it is possible to choose S so that the γ_a can be identified with the γ'_k of (4.1) (with $q = 0$) for $a = k = 1, 2, \ldots, 2n$. The real Clifford algebra generated by the γ_a is then $\mathscr{C}(2n, 0)$ and, according to (4.6),

$$\mathscr{C}(2n, 0) \simeq \begin{cases} \mathscr{M}(2^n, \mathrm{R}) & n = 0 \text{ or } 1 \pmod 4 \\ \mathscr{M}(2^{n-1}, \mathrm{H}) & n = 2 \text{ or } 3 \pmod 4. \end{cases} \tag{4.15}$$

Again the γ_a intertwine the two fundamental spinor irreps of so*$(2n)$. However the $[\gamma_a, \gamma_b]$ now span so$(2n)$ rather than so*$(2n)$.

The source of these peculiarities lies, on the one hand, in the fact that the defining representation of so*$(2n)$ is not real, so there is no *natural* choice for the signature of the matrix θ in (4.12), and on the other, in the fact that, once a real θ has been chosen, the operators γ_a and $[\gamma_a, \gamma_b]$ together span a real Lie algebra which is some real form of (the complexification of) so$(2n+1)$, as a consequence of (4.12). In general, none of these real forms contains so*$(2n)$ as a subalgebra spanned by the $[\gamma_a, \gamma_b]$. (Roughly speaking, there is no 'so*$(2n+1)$'.)

IV.6. The reader might conclude at this point that there is no real Clifford algebra unambiguously associated with either of the two fundamental spinor irreps of so*$(2n)$, or with their direct sum. However, this is not the case. For the direct sum, we consider the real algebra $\mathscr{C}^*_{2n}$ generated under multiplication and real linear combination by the so*$(2n)$ operators $\mathrm{i}L_{rs}$, $\mathrm{i}M_{rs}$ etc. This is the analogue of the even part $\mathscr{C}^+(p, q)$ of the Clifford algebra $\mathscr{C}^+(p, q)$ pertaining to the case of so(p, q), but obviously is defined independently of any choice of Dirac matrices.

It is not immediately clear whether or not $\mathscr{C}^*_{2n}$ can be regarded as a Clifford algebra. For example, consideration of the triangular array (3.3) for $n = 3$ shows that we are then considering the algebra generated by $i\sigma_3$, $i\sigma_3\rho_3$, $i\rho_3\tau_3$, $i(\sigma_2-\sigma_2\rho_3)$, $(\sigma_2+\sigma_2\rho_3)$ etc, and these do not suggest any obvious set of generators satisfying elementary anticommutation relations.

For general n, we begin by identifying the central element ε of $\mathscr{C}^*_{2n}$ as

$$\varepsilon=(iM_{11})(iM_{22})\ldots(iM_{nn}). \tag{4.16}$$

We see from (2.23) and (3.3) that, in terms of Pauli matrices,

$$\varepsilon=(i)^n\omega_3 \qquad \varepsilon^2=(-1)^nI \tag{4.17}$$

where ω_α form the last set of Pauli matrices used to construct the triangular array for so($2n$) (and so($2n+1$)); thus ε is the same operator ε' as constructed in IV.1 (with $q=0$).

IV.7. Suppose now that n is odd, so that $\varepsilon^2=-I$. Then set

$$\begin{aligned}\zeta_{r-1}&=i(L_{1r}-Q_{1r}\varepsilon) \qquad r=2,3,\ldots,n\\ \zeta_{n+r-1}&=i(M_{1r}-P_{1r}\varepsilon) \qquad r=1,2,\ldots,n\end{aligned} \tag{4.18}$$

and note that each is an element of $\mathscr{C}^*_{2n}$. In terms of ω_3 and the so($2n$) operators J_{kl} we have

$$\begin{aligned}\zeta_{r-1}&=2i(-1)^{r+1}\{J_{1\,2r-1}\tfrac{1}{2}(I+\omega)+J_{2\,2r}\tfrac{1}{2}(I-\omega)\} \qquad r\neq 1\\ \zeta_{n+r-1}&=2i(-1)^{r+1}\{J_{1\,2r}\tfrac{1}{2}(I+\omega)-J_{2\,2r-1}\tfrac{1}{2}(I-\omega)\}\end{aligned} \tag{4.19}$$

for $r=1, 2, \ldots, n$, where

$$\omega=(-1)^{(n-1)/2}\omega_3 \qquad \omega^2=I \tag{4.20}$$

and, since ω_3 commutes with J_{1k} and J_{2k}, we see from (4.19) that

$$\{\zeta_P,\zeta_Q\}=-2\delta_{PQ} \qquad P,Q=1,2,\ldots,2n-1. \tag{4.21}$$

These ζ_P therefore generate a real Clifford algebra [24, 10, 11] $\mathscr{C}(0, 2n-1)\simeq \mathscr{M}(2^{n-1}, C)$, (since n is odd), which is a subalgebra of $\mathscr{C}^*_{2n}$, by construction. The dimension of this real subalgebra is $2^{n-1}\times 2^{n-1}\times 2$; but this is an upper bound on the dimension of $\mathscr{C}^*_{2n}$ itself, given that the L_{rs} etc are associated with the direct sum of two 2^{2n-1}-dimensional irreps. It follows that the ξ_P generate the whole of $\mathscr{C}^*_{2n}$, and that

$$\mathscr{C}^*_{2n}\simeq\mathscr{C}(0,2n-1)\simeq\mathscr{M}(2^{n-1},\mathrm{C}) \qquad n \text{ odd}. \tag{4.22}$$

As remarked in IV.2, this algebra is simple, with centre isomorphic to C; the centre is spanned by I and ε. Consistent with (4.22) is the fact that, when n is odd, neither of the fundamental spinor irreps of so*($2n$) is real; a consideration of their weights, which are the same as those of the corresponding irreps of so($2n$), shows that they transform into each other under complex conjugation.

IV.8. Suppose now that $n=2m$ is even. In this case, from (4.17),

$$\varepsilon=(-1)^m\omega_3 \qquad \varepsilon^2=I.$$

Then $\mathscr{C}^*_{2n}$ is not simple, but splits into a (direct) sum of two ideals,

$$\mathscr{C}^*_{2n}=\mathscr{C}^{*(+)}_{2n}\oplus\mathscr{C}^{*(-)}_{2n} \qquad \mathscr{C}^{*(\pm)}_{2n}=\mathscr{C}^*_{2n}\tfrac{1}{2}(I\pm\varepsilon). \tag{4.23}$$

We note also from (3.17) that

$$\eta=(-i)^m\zeta=(I+iM_{11})(I+iM_{22})\ldots(I+iM_{nn})/2^m \tag{4.24}$$

is in $\mathscr{C}^*_{2n}$. The two ideals in (4.23) are associated with the eigenvalues $\pm(-1)^m$ of ω_3, and hence with the irreps $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \pm(-1)^m\frac{1}{2})$ of so*$(2n)$. In $\mathscr{C}^{*(+)}_{2n}$, ω_3 can be replaced by $(-1)^m I$, and according to the results of III.6, $\zeta^2 = (-1)^m$ on the corresponding subspace. It follows that $\eta^2 = I$ on $\mathscr{C}^{*(+)}_{2n}$; furthermore, according to (3.14),

$$[\eta, L_{rs}] = [\eta, M_{rs}] = \{\eta, P_{rs}\} = \{\eta, Q_{rs}\} = 0. \tag{4.25}$$

In $\mathscr{C}^{*(+)}_{2n}$, we now set

$$\varphi_{r-1} = \mathrm{i}(M_{1r} - \eta P_{1r}) \qquad \varphi_{n-2+r} = \mathrm{i}(L_{1r} - \eta Q_{1r}) \tag{4.26}$$

for $r = 2, 3, \ldots, n$. It can then be checked that these satisfy

$$\{\varphi_A, \varphi_B\} = -2\delta_{AB} \qquad A, B = 1, 2, \ldots, 2n-2. \tag{4.27}$$

For example

$$\begin{aligned}\{\varphi_1, \varphi_{n+2}\} &= -\{M_{12}, L_{14}\} + \{P_{12}, Q_{14}\} + \eta\{M_{12}, Q_{14}\} + \eta\{P_{12}, L_{14}\} \\ &\qquad \text{(using the properties of } \eta) \\ &= -\{J_{14} - J_{23}, J_{17} + J_{28}\} - \{J_{14} + J_{23}, J_{17} - J_{28}\} \\ &\quad + \mathrm{i}\eta\{J_{14} - J_{23}, J_{17} - J_{28}\} + \mathrm{i}\eta\{J_{14} + J_{23}, J_{17} + J_{28}\}\end{aligned} \tag{4.28}$$

which vanishes because $\{J_{14}, J_{17}\}$ and $\{J_{23}, J_{28}\}$ vanish.

The φ_A generate a real Clifford algebra $\mathscr{C}(0, 2n-2)$, which has dimension [10, 11] $2^{n-1} \times 2^{n-1}$. Since this is an upper bound for the dimension of $\mathscr{C}^{*(+)}_{2n}$, we have

$$\mathscr{C}^{*(+)}_{2n} \simeq \mathscr{C}(0, 2n-2) \simeq \begin{cases} \mathscr{M}(2^{n-2}, \mathrm{H}) & m \text{ odd} \\ \mathscr{M}(2^{n-1}, \mathrm{R}) & m \text{ even.} \end{cases} \tag{4.29}$$

Turning now to $\mathscr{C}^{*(-)}_{2n}$, where ω_3 can be replaced by $(-1)^{m+1}I$, we have $\eta^2 = -I$. We introduce

$$\xi = (\mathrm{i}M_{11})\eta \tag{4.30}$$

which is in $\mathscr{C}^*_{2n}$ and satisfies $\xi^2 = I$ in $\mathscr{C}^{*(-)}_{2n}$. Then

$$\{\xi, L_{1r}\} = \{\xi, M_{1r}\} = [\xi, P_{1r}] = [\xi, Q_{1r}] = 0 \tag{4.31}$$

provided $r \neq 1$. In $\mathscr{C}^{*(-)}_{2n}$, we set

$$\psi_{r-1} = \mathrm{i}Q_{1r} + \mathrm{i}\xi L_{1r} \qquad \psi_{n-2+r} = \mathrm{i}P_{1r} + \mathrm{i}\xi M_{1r} \tag{4.32}$$

for $r = 2, 3, \ldots, n$, and find that these satisfy

$$\{\psi_A, \psi_B\} = 2\delta_{AB} \qquad A, B = 1, 2, \ldots, 2n-2. \tag{4.33}$$

Then we deduce that

$$\mathscr{C}^{*(-)}_{2n} \simeq \mathscr{C}(2n-2, 0) \simeq \begin{cases} \mathscr{M}(2^{n-1}, \mathrm{R}) & m \text{ odd} \\ \mathscr{M}(2^{n-2}, \mathrm{H}) & m \text{ even.} \end{cases} \tag{4.34}$$

Combining the results (4.29) and (4.34), we see that (cf (4.9))

$$\mathscr{C}^*_{2n} \simeq \mathscr{C}(0, 2n-2) \oplus \mathscr{C}(2n-2, 0) \simeq \mathscr{M}(2^{n-1}, \mathrm{R}) \oplus \mathscr{M}(2^{n-2}, \mathrm{H}) \qquad n \text{ even.} \tag{4.35}$$

This algebra has centre isomorphic to $\mathrm{R} \oplus \mathrm{R}$, generated by I and ε. The subalgebra $\mathscr{M}(2^{n-1}, \mathrm{R})$ is associated with the value 1 of ω_3 and $(-1)^m$ of ε, and hence with the irrep $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})$ of so*$(2n)$, which is therefore real. The subalgebra $\mathscr{M}(2^{n-2}, \mathrm{H})$ is associated with the values -1 and $(-1)^{m+1}$ of ω_3 and ε, and with the irrep $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$, which is therefore complex (though pseudo-real). Thus there is a remarkable asymmetry between the two fundamental spinor irreps of so*$(4n)$.

V. The isomorphism between so*(6) and su(3, 1)

V.1. Given basis elements l_{rs}, m_{rs}, p_{rs} and q_{rs} of so*(6), as in (2.16), with $r, s = 1, 2, 3$, we set

$$\begin{aligned} &\Lambda_{rs} = l_{rs} \qquad \Lambda_{r4} = -\Lambda_{4r} = \tfrac{1}{2}\varepsilon_{rst}q_{st} \\ &M_{rs} = m_{rs} - \tfrac{1}{2}\delta_{rs}m_{uu} \qquad M_{44} = -\tfrac{1}{2}m_{uu} \\ &M_{4r} = M_{r4} = \tfrac{1}{2}\varepsilon_{rst}p_{st}. \end{aligned} \tag{5.1}$$

Then we find that, for $\mu, \nu, \rho, \sigma = 1, 2, 3, 4$,

$$\begin{aligned} &[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}(g_{\nu\rho}\Lambda_{\mu\sigma} - g_{\mu\rho}\Lambda_{\sigma\nu} - g_{\mu\sigma}\Lambda_{\rho\nu} + g_{\nu\sigma}\Lambda_{\mu\rho}) \\ &[\Lambda_{\mu\nu}, X_{\rho\sigma}] = \mathrm{i}(-g_{\nu\rho}X_{\mu\sigma} + g_{\mu\rho}X_{\nu\sigma} + g_{\mu\sigma}X_{\rho\nu} - g_{\nu\sigma}X_{\rho\mu}) \end{aligned} \tag{5.2}$$

where $X_{\rho\sigma}$ represents either of $M_{\rho\sigma}$, $\Lambda_{\rho\sigma}$, and where $(g_{\mu\nu}) = \mathrm{diag}(1, 1, 1, -1)$. Furthermore

$$g^{\mu\nu}M_{\mu\nu} = 0 \tag{5.3}$$

where $g^{\mu\nu} = g_{\mu\nu}$. These are the defining relations of su(3, 1), in a standard form (cf (2.17)), and since the linear transformation (5.1) from the basis elements of so*(6) to those of su(3, 1) is real and invertible, this establishes the isomorphism of the two real Lie algebras.

V.2. A finite-dimensional irrep of su(3, 1) is labelled [25] by a highest weight defined with respect to the Cartan subalgebra $(M_{11} + \tfrac{1}{2}M_{22} + \tfrac{1}{2}M_{33},\ M_{22} + \tfrac{1}{2}M_{33} + \tfrac{1}{2}M_{11},\ M_{33} + \tfrac{1}{2}M_{11} + \tfrac{1}{2}M_{22})$; we find that the irrep (m_1, m_2, m_3) of so*(6) is mapped by (5.1) into the irrep $(m_1 + m_2, m_1 - m_3, m_2 - m_3)$ of su(3, 1).

In section VII, we have taken the basis elements of the su(3)⊕u(1) subalgebra of su(3, 1) to be Λ_{rs} $(= l_{rs})$, $M_{rs} - \tfrac{1}{3}\delta_{rs}M_{uu}$ $(= m_{rs} - \tfrac{1}{3}\delta_{rs}m_{uu})$, and $\tfrac{2}{3}M_{44}$ $(= -\tfrac{1}{3}M_{uu})$; the scaling of the u(1) element is of course at our disposal, and we have chosen it so that the su(3) triplet contained in the four-dimensional irrep (1, 0, 0) of su(3, 1) has u(1) value (*baryon number*) 1/3.

VI. The isomorphism between so*(8) and so(6, 2)

VI.1. The existence of this isomorphism [26] is closely related to the triality property [4, 9, 10] of so(8). If we set

$$\begin{aligned} &k_{12} = \tfrac{1}{4}(m_{11} - m_{22} + m_{33} - m_{44}) = \tfrac{1}{2}(j_{12} - j_{34} + j_{56} - j_{78}) \\ &k_{34} = \tfrac{1}{4}(m_{11} - m_{22} - m_{33} + m_{44}) = \tfrac{1}{2}(j_{12} - j_{34} - j_{56} + j_{78}) \\ &k_{56} = \tfrac{1}{4}(m_{11} + m_{22} - m_{33} - m_{44}) = \tfrac{1}{2}(j_{12} + j_{34} - j_{56} - j_{78}) \\ &k_{78} = \tfrac{1}{4}(m_{11} + m_{22} + m_{33} + m_{44}) = \tfrac{1}{2}(j_{12} + j_{34} + j_{56} + j_{78}) \end{aligned} \tag{6.1a}$$

$$\begin{aligned} &k_{13} = -\tfrac{1}{2}(l_{12} + l_{34}) = \tfrac{1}{2}(j_{13} + j_{24} + j_{57} + j_{68}) \\ &k_{14} = \tfrac{1}{2}(m_{12} - m_{34}) = \tfrac{1}{2}(j_{23} - j_{14} - j_{67} + j_{58}) \\ &k_{15} = -\tfrac{1}{2}(m_{14} + m_{23}) = -\tfrac{1}{2}(j_{27} - j_{18} + j_{45} - j_{36}) \\ &k_{16} = -\tfrac{1}{2}(l_{14} + l_{23}) = \tfrac{1}{2}(j_{17} + j_{28} + j_{35} + j_{46}) \\ &k_{23} = \tfrac{1}{2}(m_{12} + m_{34}) = \tfrac{1}{2}(j_{23} - j_{14} + j_{67} - j_{58}) \end{aligned}$$

$$
\begin{aligned}
k_{24} &= \tfrac{1}{2}(l_{12} - l_{34}) = -\tfrac{1}{2}(j_{13} + j_{24} - j_{57} - j_{68}) \\
k_{25} &= \tfrac{1}{2}(l_{23} - l_{14}) = \tfrac{1}{2}(j_{17} + j_{28} - j_{35} - j_{46}) \\
k_{26} &= \tfrac{1}{2}(m_{14} - m_{23}) = \tfrac{1}{2}(j_{27} - j_{18} - j_{45} + j_{36}) \\
k_{35} &= \tfrac{1}{2}(m_{24} - m_{13}) = \tfrac{1}{2}(j_{25} - j_{16} - j_{47} + j_{38}) \\
k_{36} &= \tfrac{1}{2}(l_{24} - l_{13}) = -\tfrac{1}{2}(j_{15} + j_{26} - j_{37} - j_{48}) \\
k_{45} &= -\tfrac{1}{2}(l_{13} + l_{24}) = -\tfrac{1}{2}(j_{15} + j_{26} + j_{37} + j_{48}) \\
k_{46} &= \tfrac{1}{2}(m_{13} + m_{24}) = -\tfrac{1}{2}(j_{25} - j_{16} + j_{47} - j_{38})
\end{aligned} \tag{6.1b}
$$

$$
\begin{aligned}
k_{17} &= \tfrac{1}{2}(p_{13} - p_{24}) = \tfrac{1}{2}\mathrm{i}(j_{25} + j_{16} - j_{47} - j_{38}) \\
k_{18} &= \tfrac{1}{2}(q_{13} - q_{24}) = \tfrac{1}{2}\mathrm{i}(j_{15} - j_{26} - j_{37} + j_{48}) \\
k_{27} &= -\tfrac{1}{2}(q_{13} + q_{24}) = -\tfrac{1}{2}\mathrm{i}(j_{15} - j_{26} + j_{37} - j_{48}) \\
k_{28} &= \tfrac{1}{2}(p_{13} + p_{24}) = \tfrac{1}{2}\mathrm{i}(j_{25} + j_{16} + j_{47} + j_{38}) \\
k_{37} &= -\tfrac{1}{2}(p_{23} + p_{14}) = \tfrac{1}{2}\mathrm{i}(j_{45} + j_{36} + j_{27} + j_{18}) \\
k_{38} &= -\tfrac{1}{2}(q_{23} + q_{14}) = \tfrac{1}{2}\mathrm{i}(j_{35} - j_{46} + j_{17} - j_{28}) \\
k_{47} &= \tfrac{1}{2}(q_{14} - q_{23}) = -\tfrac{1}{2}\mathrm{i}(j_{17} - j_{28} - j_{35} + j_{46}) \\
k_{48} &= \tfrac{1}{2}(p_{23} - p_{14}) = \tfrac{1}{2}\mathrm{i}(j_{27} + j_{18} - j_{45} - j_{36}) \\
k_{57} &= \tfrac{1}{2}(q_{12} - q_{34}) = -\tfrac{1}{2}\mathrm{i}(j_{13} - j_{24} - j_{57} + j_{68}) \\
k_{58} &= \tfrac{1}{2}(p_{34} - p_{12}) = \tfrac{1}{2}\mathrm{i}(j_{23} + j_{14} - j_{67} - j_{58}) \\
k_{67} &= \tfrac{1}{2}(p_{12} + p_{34}) = -\tfrac{1}{2}\mathrm{i}(j_{23} + j_{14} + j_{67} + j_{58}) \\
k_{68} &= \tfrac{1}{2}(q_{12} + q_{34}) = -\tfrac{1}{2}\mathrm{i}(j_{13} - j_{24} + j_{57} - j_{68})
\end{aligned} \tag{6.1c}
$$

we find that the k_{CD} $(=-k_{DC})$, $C, D = 1, 2, \ldots, 8$, satisfy the defining relations of so(6, 2):

$$[k_{CD}, k_{EF}] = \mathrm{i}(g_{CE}k_{DF} + g_{DF}k_{CE} - g_{DE}k_{CF} - g_{CF}k_{DE}) \tag{6.2}$$

where $(g_{CD}) = \mathrm{diag}(1, 1, 1, 1, 1, 1, -1, -1)$. Since the linear relations (6.1) between the k_{CD} and the l_{rs}, m_{rs} etc. are real and invertible, this establishes the isomorphism in a direct if not very illuminating way. It can be given a deeper interpretation by considering the algebras involved as algebras over the quaternions [20]. Note that the k_{CD} for $C, D = 1, 2, \ldots, 6$ and k_{78} involve only the l_{rs} and m_{rs}, and vice versa. These are associated with $\mathrm{so}(6) \oplus \mathrm{so}(2) < \mathrm{so}(6, 2)$ on the one hand, and $\mathrm{u}(4) < \mathrm{so}^*(8)$ on the other. (Recall that $\mathrm{so}(6) \simeq \mathrm{su}(4)$.)

VI.2. Note that this mapping between so*(8) and so(6, 2) can transform a spinor (or tensor) representation of the one into a spinor or tensor representation of the other. We need only observe that when the eigenvalues of J_{12}, J_{34}, J_{56}, and J_{78} are all half-integral, those of K_{12}, K_{34}, K_{56} and K_{78} defined by analogy to (6.1a) can be either all integral or all half-integral, and the same is true if the eigenvalues of the former set are all integral.

Evidently an irrep of so*(8) is mapped into an irrep of so(6, 2) and vice versa. The question arises: given the finite-dimensional irrep (m_1, m_2, m_3, m_4) of so*(8), into what irrep of so(6, 2) is it mapped?

To answer this, we note that when $(J_{12}, J_{34}, J_{56}, J_{78})$ have the eigenvalues (m_1, m_2, m_3, m_4) in this irrep, the corresponding eigenvector corresponds to a highest weight and so must be annihilated by the *raising operators* $J_{13}+iJ_{23}, J_{14}+iJ_{24}, \ldots; J_{35}+iJ_{45}, J_{36}+iJ_{46}, \ldots; J_{57}+iJ_{67}, J_{58}+iJ_{68}$. But then it follows from (6.1) that this vector is annihilated also by the operators $K_{76}-iK_{86}, K_{75}-iK_{85}, \ldots; K_{54}+iK_{64}, K_{53}+iK_{63}, \ldots; K_{13}+iK_{23}, K_{14}+iK_{24}$; so that it also corresponds to a highest weight in so(6, 2) with respect to the Cartan subalgebra $(K_{78}, K_{56}, K_{12}, K_{34})$. The corresponding weight vector in so(6, 2) is seen from (6.1*a*) to be (m_1', m_2', m_3', m_4'), where

$$\begin{aligned} m_1' &= \tfrac{1}{2}(m_1+m_2+m_3+m_4) \\ m_2' &= \tfrac{1}{2}(m_1+m_2-m_3-m_4) \\ m_3' &= \tfrac{1}{2}(m_1-m_2+m_3-m_4) \\ m_4' &= \tfrac{1}{2}(m_1-m_2-m_3+m_4). \end{aligned} \tag{6.3}$$

Thus the mapping (6.1) carries the irrep (m_1, m_2, m_3, m_4) of so*(8) into the irrep (m_1', m_2', m_3', m_4') of so(6, 2). (Note that it therefore carries also the irrep (m_1', m_2', m_3', m_4') of so*(8) into the irrep (m_1, m_2, m_3, m_4) of so(6, 2).)

For example, the spinor irrep $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ of so*(8) is mapped into the tensor irrep (1, 1, 1, 0) of so(6, 2), while the spinor irrep $(\frac{7}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2})$ is mapped into the spinor irrep $(\frac{5}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$.

VI.3. There is a check by dimensions [7]:

$$\begin{aligned} d(m_1, m_2, m_3, m_4) &= d(m_1', m_2', m_3', m_4') \\ &= (m_1-m_2+1)(m_1+m_2+5)(m_1-m_3+2)(m_1+m_3+4) \\ &\quad \times(m_1-m_4+3)(m_1+m_4+3)(m_2-m_3+1)(m_2+m_3+3) \\ &\quad \times(m_2-m_4+2)(m_2+m_4+2)(m_3-m_4+1) \\ &\quad \times(m_3+m_4+1)/5!3!3!. \end{aligned} \tag{6.4}$$

VI.4. Of particular interest are the defining 8-vector irrep (1, 0, 0, 0); the two 8-spinor irreps $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$; and the 28-dimensional adjoint irrep (1, 1, 0, 0) of so*(8). These are mapped into the so(6, 2) irreps $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, (1, 0, 0, 0), $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ and (1, 1, 0, 0), respectively.

The interchange of the 8-dimensional vector and spinor irreps (1, 0, 0, 0) and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ can be regarded as a relic of the triality property of so(8), whereby isomorphisms exist which interchange any two of the three 8-dimensional irreps of that algebra, leaving the third invariant [9, 10, 4]. This can be seen as follows.

If we replace $k_{17}, k_{18}, \ldots, k_{68}$ in (6.1*c*) by ik_{17}, ik_{18} etc, and k_{78} in (6.1*a*) by $-k_{78}$, the operators so defined, together with the old k_{CD} for $C, D=1, 2, \ldots, 6$, satisfy the defining relations of so(8) rather than so(6, 2), i.e. relations like (6.2) with g_{CD} replaced by δ_{CD}. Furthermore, they are then related to the so(8) operators j_{CD} by a real, invertible linear transformation. Then we have an isomorphism from so(8) into so(8) that interchanges one of the 8-spinor irreps with the 8-vector irrep, leaving the other 8-spinor irrep invariant. However, we have also the alternative possibility of replacing in (6.1*c*) $k_{17}, k_{18}, k_{27}, \ldots$ by $-ik_{17}, +ik_{18}, -ik_{27}, \ldots$ and leaving k_{78} in (6.1*a*) unchanged. Then we again have an isomorphism from so(8) into so(8), this time interchanging the second 8-spinor irrep with the 8-vector irrep, and leaving the first invariant. These two isomorphisms—together with the isomorphism (whose analogue exists for so($2n$), any n) that replaces j_{C8} by $-j_{C8}$, $C=1, 2, \ldots, 7$ and so interchanges the two 8-spinor irreps, leaving the 8-vector invariant—define the triality property of so(8).

A necessary condition for the triality is that the three 8-dimensional irreps have similar reality properties; in fact, for so(8) all three are real. As we have seen in IV.8, the 8-spinor irrep $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ of so*(8) is real, while $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ is pseudo-real, as is the 8-vector defining irrep (1, 0, 0, 0). For so(6, 2), both of the 8-spinor irreps are pseudo-real, as we saw in IV.3, while the defining representation is real. Of the triality isomorphisms, only one mapping survives when we go from so(8) to its complexification and thence to the real forms so*(8) and so(6, 2)—the mapping (6.1), which maps the real 8-spinor irrep, the pseudo-real 8-spinor irrep, and the pseudo-real 8-vector irrep of so*(8) into the real 8-vector irrep and the two pseudo-real 8-spinor irreps, respectively, of so(6, 2).

Is there a version of the triality property involving any other of the real forms so(p, q), with $p+q=8$, $p \geq q$? There are no isomorphisms between these algebras, and only for so(4, 4) and so(8) do the defining irrep and the two fundamental spinor irreps have the same reality properties, as can be seen from the results of IV. It seems therefore that, apart from the so(8) case, and that involving so*(8) and so(6, 2), a form of triality can hold only for [27] so(4, 4).

VII. Physical applications

VII.1. It is primarily the isomorphism of so*(8) with so(6, 2) that gives rise to interesting possible applications of so*($2n$) ($n \geq 4$) as an extended spacetime symmetry algebra. We have already indicated in section I, some of Ward's ideas [19] concerning so*(14). In this connection we consider only the reduction with respect to so(3, 1)⊕su(3)⊕su(2)⊕u(1) of the two fundamental spinor irreps, each of dimension 64; the defining 14-vector irrep; and the 91-dimensional adjoint irrep.

Firstly, with respect to the reduction so*(14) > so*(8)⊕so*(6) we have (the branching rules are as for [6] so(14) > so(8)⊕so(6); we show the dimensions of irreps beneath their labels)

$$\begin{gathered}
\underset{(64)}{(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\pm\tfrac{1}{2})} = [\underset{(8)}{(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\pm\tfrac{1}{2})}, \underset{(4)}{(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})}] \oplus [\underset{(8)}{(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\mp\tfrac{1}{2})}, \underset{(4)}{(\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})}] \\
\underset{(14)}{(1,0,0,0,0,0,0)} = [\underset{(8)}{(1,0,0,0)}, \underset{(1)}{(0,0,0)}] \oplus [\underset{(1)}{(0,0,0,0)}, \underset{(6)}{(1,0,0)}] \\
\underset{(91)}{(1,1,0,0,0,0,0)} = [\underset{(28)}{(1,1,0,0)}, \underset{(1)}{(0,0,0)}] \oplus [\underset{(8)}{(1,0,0,0)}, \underset{(6)}{(1,0,0)}] \oplus [\underset{(1)}{(0,0,0,0)}, \underset{(15)}{(1,1,0)}].
\end{gathered} \tag{7.1}$$

Next we map each so*(8) irrep into a corresponding irrep of so(6, 2) according to (6.3), and reduce it with respect to so(3, 2)⊕su(2) according to well known rules. Also, we map each so*(6) irrep into a corresponding irrep of su(3, 1) according to (5.1), and reduce with respect to su(3)⊕u(1). For example, when so*(8)⊕so*(6)→so(6, 2)⊕su(3, 1),

$$[(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}), (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})] \rightarrow [(1,0,0,0), (1,0,0)] \tag{7.2}$$

and then, with respect to $\mathrm{so}(6,2) > \mathrm{so}(3,2) \oplus \mathrm{su}(2)$

$$\underset{(8)}{(1,0,0,0)} = [\underset{(5)}{(1,0)}, \underset{(1)}{(0)}] \oplus [\underset{(1)}{(0,0)} \underset{(3)}{(1)}] \tag{7.3}$$

while for $\mathrm{su}(3,1) > \mathrm{su}(3) \oplus \mathrm{u}(1)$ we have

$$\underset{(4)}{(1,0,0)} = [\underset{(3)}{(1,0)}, \underset{(1)}{(\tfrac{1}{3})}] \oplus [\underset{(1)}{(0,0)}, \underset{(1)}{(-1)}]. \tag{7.4}$$

In this way we find that, with respect to $\mathrm{so}^*(14) > \mathrm{so}(3,2) \oplus \mathrm{su}(3) \oplus \mathrm{su}(2) \oplus \mathrm{u}(1)$ we have

$$\underset{(64)}{(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})} = [\underset{(5)}{(1,0)}, \underset{(3)}{(1,0)}, \underset{(1)}{(0)}, \underset{(1)}{(\tfrac{1}{3})}] \oplus [\underset{(5)}{(1,0)}, \underset{(1)}{(0,0)}, \underset{(1)}{(0)}, \underset{(1)}{(-1)}]$$

$$\oplus [\underset{(1)}{(0,0)}, \underset{(3)}{(1,0)}, \underset{(3)}{(1)}, \underset{(1)}{(\tfrac{1}{3})}] \oplus [\underset{(1)}{(0,0)}, \underset{(1)}{(0,0)}, \underset{(3)}{(1)}, \underset{(1)}{(-1)}]$$

$$\oplus [\underset{(4)}{(\tfrac{1}{2},\tfrac{1}{2})}, \underset{(\bar{3})}{(1,1)}, \underset{(2)}{(\tfrac{1}{2})}, \underset{(1)}{(-\tfrac{1}{3})}] \oplus [\underset{(4)}{(\tfrac{1}{2},\tfrac{1}{2})}, \underset{(1)}{(0,0)}, \underset{(2)}{(\tfrac{1}{2})}, \underset{(1)}{(1)}]$$

$$\underset{(64)}{(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})} = [\underset{(5)}{(1,0)}, \underset{(\bar{3})}{(1,1)}, \underset{(1)}{(0)}, \underset{(1)}{(-\tfrac{1}{3})}] \oplus [\underset{(5)}{(1,0)}, \underset{(1)}{(0,0)}, \underset{(1)}{(0)}, \underset{(1)}{(1)}]$$

$$\oplus [\underset{(1)}{(0,0)}, \underset{(\bar{3})}{(1,1)}, \underset{(3)}{(1)}, \underset{(1)}{(-\tfrac{1}{3})}] \oplus [\underset{(1)}{(0,0)}, \underset{(1)}{(0,0)}, \underset{(3)}{(1)}, \underset{(1)}{(1)}]$$

$$\oplus [\underset{(4)}{(\tfrac{1}{2},\tfrac{1}{2})}, \underset{(3)}{(1,0)}, \underset{(2)}{(\tfrac{1}{2})}, \underset{(1)}{(\tfrac{1}{3})}] \oplus [\underset{(4)}{(\tfrac{1}{2},\tfrac{1}{2})}, \underset{(1)}{(0,0)}, \underset{(2)}{(\tfrac{1}{2})}, \underset{(1)}{(-1)}] \tag{7.5}$$

$$\underset{(14)}{(1,0,0,0,0,0,0)} = [\underset{(4)}{(\tfrac{1}{2},\tfrac{1}{2})}, \underset{(1)}{(0,0)}, \underset{(2)}{(\tfrac{1}{2})}, \underset{(1)}{(0)}] \oplus [\underset{(1)}{(0,0)}, \underset{(3)}{(1,0)}, \underset{(1)}{(0)}, \underset{(1)}{(-\tfrac{2}{3})}]$$

$$\oplus [\underset{(1)}{(0,0)}, \underset{(\bar{3})}{(1,1)}, \underset{(1)}{(0)}, \underset{(1)}{(\tfrac{2}{3})}]$$

$$\underset{(91)}{(1,1,0,0,0,0,0)} = [\underset{(10)}{(1,1)}, \underset{(1)}{(0,0)}, \underset{(1)}{(0)}, \underset{(1)}{(0)}] \oplus [\underset{(5)}{(1,0)}, \underset{(1)}{(0,0)}, \underset{(3)}{(1)}, \underset{(1)}{(0)}]$$

$$\oplus [\underset{(1)}{(0,0)}, \underset{(1)}{(0,0)}, \underset{(3)}{(1)}, \underset{(1)}{(0)}] \oplus [\underset{(1)}{(0,0)}, \underset{(\bar{3})}{(1,1)}, \underset{(1)}{(0)}, \underset{(1)}{(-\tfrac{4}{3})}]$$

$$\oplus [\underset{(1)}{(0,0)}, \underset{(8)}{(2,1)}, \underset{(1)}{(0)}, \underset{(1)}{(0)}] \oplus [\underset{(1)}{(0,0)}, \underset{(1)}{(0,0)}, \underset{(1)}{(0)}, \underset{(1)}{(0)}]$$

$$\oplus [\underset{(1)}{(0,0)}, \underset{(3)}{(1,0)}, \underset{(1)}{(0)}, \underset{(1)}{(\tfrac{4}{3})}] \oplus [\underset{(4)}{(\tfrac{1}{2},\tfrac{1}{2})}, \underset{(3)}{(1,0)}, \underset{(2)}{(\tfrac{1}{2})}, \underset{(1)}{(-\tfrac{2}{3})}]$$

$$\oplus [\underset{(4)}{(\tfrac{1}{2},\tfrac{1}{2})}, \underset{(\bar{3})}{(1,1)}, \underset{(2)}{(\tfrac{1}{2})}, \underset{(1)}{(\tfrac{2}{3})}].$$

Here the irreps $(1,1)$, $(1,0)$, $(\frac{1}{2},\frac{1}{2})$ and $(0,0)$ of so(3, 2) are the 10-tensor (adjoint), 5-vector, 4-spinor and scalar, respectively. Their reduction with respect to so(3, 1) < so(3, 2) is well known:

$$\begin{array}{l}
\underset{(10)}{(1,1)} = \underset{(3)}{(1,1)} \oplus \underset{(4)}{(1,0)} \oplus \underset{(3)}{(1,-1)} \\
\underset{(5)}{(1,0)} = \underset{(4)}{(1,0)} \oplus \underset{(1)}{(0,0)} \\
\underset{(4)}{(\frac{1}{2},\frac{1}{2})} = \underset{(2)}{(\frac{1}{2},\frac{1}{2})} \oplus \underset{(2)}{(\frac{1}{2},-\frac{1}{2})} \\
\underset{(1)}{(0,0)} = \underset{(1)}{(0,0)}
\end{array} \tag{7.6}$$

where $(1,0)$ is the 4-vector, $(\frac{1}{2},\frac{1}{2})\oplus(\frac{1}{2},-\frac{1}{2})$ the Dirac spinor, and $(1,1)\oplus(1,-1)$ is the antisymmetric tensor (adjoint) representation.

VII.2. We shall not pursue the interpretation of these branchings in a field theory based on so*(14), merely noting that, as expected from our earlier remarks, both fermions and bosons can be accommodated in a single irreducible so*(14) multiplet. Instead, we make a few remarks about a different, but related idea. In earlier work [28], the Clifford algebra $\mathscr{C}(2,5)$ has been used in a unified description of four Dirac particles, interpreted as the electron and its neutrino, the neutron and the proton. Their allowed quantum states have been associated with projectors in the algebra rather than eigenvectors of operators in a representation space. We now have the interesting possibility to extend this idea by choosing, for example, the Clifford algebra associated with one of the fundamental spinor irreps of so*($2n$), $n>4$. This gives rise to a bigger structure, able to incorporate both fermions and bosons. For example, consider the algebra $\mathscr{C}^*_{10}\simeq\mathscr{C}(0,9)$ associated with the 16-dimensional spinor irreps $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\pm\frac{1}{2})$ of so*(10), and note that

$$\mathrm{so}^*(10) > \mathrm{so}^*(8)\oplus\mathrm{so}^*(2) \simeq \mathrm{so}(6,2)\oplus\mathrm{u}(1) > \mathrm{so}(3,2)\oplus\mathrm{su}(2)\oplus\mathrm{u}(1).$$

The corresponding reduction formulae are

$$\underset{(16)}{(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\pm\tfrac{1}{2})} = [\underset{(5)}{(1,0)},\underset{(1)}{(0)},\underset{(1)}{(\pm\tfrac{1}{2})}]\oplus[\underset{(1)}{(0,0)},\underset{(3)}{(1)},\underset{(1)}{(\pm\tfrac{1}{2})}]\oplus[\underset{(4)}{(\tfrac{1}{2},\tfrac{1}{2})},\underset{(2)}{(\tfrac{1}{2})},\underset{(1)}{(\mp\tfrac{1}{2})}]. \tag{7.7}$$

We see that, associated with each so*(10) spinor is an so(3, 2) 5-vector which is an su(2) singlet; an su(2) triplet of 5-scalars; and an su(2) doublet of 4-spinors. It may be possible to associate the Lorentz 4-vector in the 5-dimensional irrep of so(3, 2) with the photon, and the two spinors with the electron and its neutrino; an interpretation of the scalars is not obvious. In any event, it is remarkable that it is apparently possible to describe configurations with spin 0 or 1, as well as spin $\frac{1}{2}$, within the framework of one Clifford algebra $\mathscr{C}^*_{10}$, by extension of the ideas of [28]. We hope to follow up these ideas elsewhere.

VII.3. Note that the occurrence of representations of so(3, 2) > so(3, 1) in the reductions of so*($2n$), $n>4$, suggests a natural role for wave equations based on so(3, 2), including Dirac's equation (that is to say, for Lubanski–Bhabha [29] equations). In the case of so*(10), however, this idea may be difficult to reconcile with the association of the 4-vector in (7.7) with the photon, since the irrep $(1,0)$ of so(3, 2) is usually associated with spinless particles in the Lubanski-Bhabha framework.

VII.4. The occurrence of both spinor and tensor irreps of so(3, 1) in an irrep of so*(10) or so*(14) (or more generally of so*($2n$), $n>4$) reminds us of supersymmetry [30], although here we are working with Lie algebras, not superalgebras. However, at least in the case of the fundamental spinor irreps of so*($2n$), which are associated with Clifford algebras containing anticommuting elements, we may well speculate that an alternative description of *supermultiplets* like those in (7.5) and (7.7), can be given in terms of some associated Lie superalgebras.

VII.5. The possibility of both spinor and tensor irreps of so(6, 2) (and hence so(3, 1)) arising can be linked to the fact that the so*($2n$) algebra for $n>4$ contains operators which transform as spinors with respect to so(6, 2), and so can change so(6, 2) representation labels by half-integral amounts. Their existence is readily understood in terms of what we have said above. Consider for example the case of so*(10). Its forty-five basis elements can be regarded as consisting of the twenty-eight basis elements of so*(8) together with two so*(8)-vector operators and an so*(8)-scalar. (The situation is comparable to that for so(10) > so(8).) However, when we map so*(8) into so(6, 2) as in (6.1), we have already seen that the 8-vector irrep of so*(8) is mapped into the 8-spinor irrep $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ of so(6, 2). Thus the basis elements of so*(10) can also be regarded as consisting of the basis elements of so(6, 2) together with two so(6, 2)-spinor operators and an so(6, 2)-scalar. Note that the two spinor operators both transform as $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and not as the pair of fundamental spinor irreps.

VII.6. In this paper, we have concentrated on finite-dimensional irreps of so*($2n$). Since so(6, 2) contains the Lie algebras of the conformal group and the Poincaré group, it would be interesting also to examine the reduction of infinite-dimensional unitary irreps of SO*($2n$), $n>4$, with respect to these subgroups—for example, the ladder representations could be analysed in this way. Both integer and half-integer spin unitary irreps of these subgroups can be expected to appear in the reduction of a single unitary irrep of SO*($2n$). In this connection one would like to find the analogue of the formula (6.3) for unitary irreps of SO*(8) and SO(6, 2).

VII.7. The groups SO(4), SO(3, 1) and SO(2, 2) are associated with 4-dimensional *spacetime* manifolds of different metric signatures. Is SO*(4) associated with some peculiar four-dimensional real spacetime? The answer is no; at least, not in the analogous way, because the 4-vector representation of SO*(4) is only pseudo-real. For the same reason, there is no direct analogue for SO*(4) of the inhomogeneous groups associated with SO(p, q). A set of four operators comprising an SO*(4)-vector cannot define, with the six generators of SO*(4), a 10-dimensional real Lie algebra. But other representations of SO*(4) may be useful in higher-dimensional Kaluza-Klein-type theories; and the study of homogeneous spaces associated with SO*($2n$), $n>4$, and of associated inhomogeneous groups, may be of interest to physics because of the isomorphism of so*(8) and so(6, 2). Finally, we remark that [13] so*(16)(>so*(14)) is a maximal subalgebra of a non-compact real form of the exceptional Lie algebra E_8, of relevance to superstring theories [4].

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